Symbol Error Rate of Space-Time Coded Multi-Antenna Wireless Cooperative Networks

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Abstract—In this paper, we propose a generalized cooperative signal transmission model for wireless networks. In this model, the source, the destination, and an arbitrary number of relay nodes may have multiple antennas. The source node uses an orthogonal space-time block code for signal transmission over its multiple antennas to each of the relays and the destination node. Each relay performs a linear processing on the received signals and transmits the processed samples to the destination after encoding them using an orthogonal space-time block code. We analyze the performance of the proposed cooperative model by finding an approximate formula for its symbol error rate at high signal to noise ratios, which determines the total diversity order of the cooperative network. The achieved results show that various cooperative structures with different numbers of relay nodes and different numbers of antennas at each of the source, the destination and the relays, can have the same error rate performance at high signal to noise ratios.

I. INTRODUCTION

The use of cooperative diversity is an effective way of countering the adverse effects of fading in wireless channels, especially when employing more than one antenna at each node of a wireless network is not applicable [1]. However, in case multiple antennas are possible, a cooperative strategy could improve the symbol error rate (SER) performance of the communication system by providing substantial additional diversity benefits. Recently, great attention has been paid to such kinds of cooperative networks, especially with applications in ad-hoc networks [2]-[6]. In [2] and [3] optimal relaying schemes and power allocation methods for multi-antenna relay networks have been investigated. The authors in [4] were more interested in the diversity-multiplexing aspects of such networks. In [5] the error rate performance of a cooperative network with single antenna source and relay nodes and a number of multi-antenna relays operating based on a decode-and-forward model, was investigated. In [6] the SER of a space-time coded cooperative model in which a double-antenna source and a single antenna relay and destination node are present, was evaluated.

In this paper, we propose a generalized cooperative model for multi-antenna wireless networks on the basis of orthogonal space-time block codes (OSTBCs). The source and the relay nodes, in this model, use OSTBCs for information transmission over their multiple antennas. The retransmitted signals from the relay nodes are obtained by linear processing of the received space-time coded signals from the source node and not by just amplifying these signals or detecting them. We analyze the performance of the proposed system by finding an approximate formula for the SER at high signal to noise ratios (SNRs) which determines the total diversity order of the system as well.

We start in the next section by describing the proposed cooperative strategy along with the system model. Section III is dedicated to the performance analysis of the system and in Section IV the simulation results are presented. We will conclude in Section V with summarizing the work.

II. THE SYSTEM MODEL

A. Space-Time Coding and Decoding

In order to describe the proposed cooperative model, we first describe the coding and decoding procedures of an OSTBC used by Node $a$ with $N_a$ transmit antennas for transmission of $B$ information symbols to Node $b$ with $N_b$ receive antennas in the assumed network. Every generated space-time code matrix in Node $a$ is a linear processing generalized orthogonal design of size $N_a \times L_a$, [7], where $L_a$ is the number of the required time slots for transmission of the elements of a code matrix over $N_a$ transmit antennas.

For transmission of a block of $B$ modulated symbols (which we call information symbols) from Node $a$ to Node $b$, they are
divided into $M_a = B/K_a$ vectors of length $K_a$, where $K_a$ is the number of information symbols per code matrix. Each vector is represented by $s_a^{(n)}$, $n = 1...M_a$, and its $i$th element is shown by $s_a^{(n,i)}$, $i = 1...K_a$. The generated code matrix corresponding to the information vector $s_a^{(n)}$, is denoted by $S_a^{(n)}$. If we assume a flat fading model for the wireless channel between two nodes, after transmission of $S_a^{(n)}$ from $a$, $b$ receives $U_{ab}^{(n)}$ as follows:

$$U_{ab}^{(n)} = \mathbf{H}_{ab}^{(n)} S_a^{(n)} + Z_{ab}^{(n)}, \quad n = 1...M_a, \quad (1)$$

where $U_{ab}^{(n)}$ is a complex valued matrix of size $N_b \times L_a$, i.e., $U_{ab}^{(n)} \in \mathbb{C}^{N_b \times L_a}$, and its $(i,j)$th element is received from the $i$th antenna of Node $b$ at the $j$th time slot. In (1), $\mathbf{H}_{ab}^{(n)} \in \mathbb{C}^{N_b \times N_a}$ is the channel coefficient matrix of $a \rightarrow b$ MIMO (Multi-Input Multi-Output) link corresponding to the $n$th code matrix whose elements are i.i.d. complex Gaussian random variables with variance $\Omega_{ab}$. These coefficients, which are assumed to be known at the receiver side, remain fixed during transmission of one code matrix, but vary independently from one code matrix to another. $Z_{ab}^{(n)} \in \mathbb{C}^{N_b \times L_a}$ is the noise matrix whose elements are assumed to be i.i.d. zero mean complex Gaussian random variables with variance $N_0$.

The information symbols of the code matrix $S_a^{(n)}$ can be detected separately after linear processing of the elements of $U_{ab}^{(n)}$ [7]. The details of the decoding procedure and linear processing procedure can be found in [8] in a matrix form. Following the outlined method in [8], we can show that by linear processing of the elements of $U_{ab}^{(n)}$, $K_a$ decision statistics are generated, each of which is a noisy version of just one transmitted information symbol, and the decision statistics used for detecting $s_a^{(n)}$ can be expressed as follows:

$$\tilde{u}_{ab}^{(n,t)} = \|\mathbf{H}_{ab}^{(n)}\|_F^2 s_a^{(n,t)} + \tilde{w}_{ab}^{(n,t)}, \quad n = 1...M_a, \quad t = 1...K_a, \quad (2)$$

where $\|\mathbf{H}_{ab}^{(n)}\|_F^2$ is the squared Frobenius norm of $\mathbf{H}_{ab}^{(n)}$ and $\tilde{w}_{ab}^{(n,t)}$ is a white zero mean Gaussian noise sample with variance $\|\mathbf{H}_{ab}^{(n)}\|_F^2 N_0$.

**B. The Proposed Cooperative Model**

Fig. 1 shows a generic model of the multi-antenna cooperative network in which $Q$ relay nodes are present. Source, destination and relay nodes are denoted as $s$, $d$, $r_q$, $q = 1...Q$, respectively, and $N_p, p \in \{s, r_1, ..., r_Q, d\}$, represents the number of antennas of the corresponding node. The source node transmits a block of $B$ information symbols to the destination and each of the relay nodes using an OSTBC designed for its number of antennas and through the same way described in the previous subsection. Here, the number of corresponding code matrices generated at the source node is specified by $M_s = B/K_s$, where $K_s$ is the number of information symbols per one code matrix of the source node (we assume that $B$ is an integer multiple of $K_s$). Upon transmission of a generated code matrix from the source node, denoted by $S_s^{(m)}$, $m = 1...M_s$, the destination receives $U_{sd}^{(m)}$ and $r_q$ receives $U_{sr_q}^{(m)}$ as shown in Table I. After the so-called linear processing of the elements of $U_{sd}^{(m)}$, the destination node generates $K_s$ decision statistics denoted by $\tilde{u}_{sr_q}^{(m,k)}$ in Table I. Similarly, $r_q$ generates $K_s$ decision statistics after the linear processing of the elements of $U_{sr_q}^{(m)}$ which are represented by $\tilde{u}_{sr_q}^{(m,k)}$ in the table.

<table>
<thead>
<tr>
<th>TABLE I</th>
<th>RECEIVED SIGNALS AND THE GENERATED DECISION STATISTICS AT THE RELAY AND DESTINATION NODES</th>
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<tbody>
<tr>
<td>Received Signals</td>
<td>Generated Decision Statistics</td>
</tr>
<tr>
<td>$U_{sd}^{(m)} = \mathbf{H}<em>{sd}^{(m)} S_s^{(m)} + Z</em>{sd}^{(m)}, m = 1...M_s$</td>
<td>$\tilde{u}<em>{sd}^{(m,k)} = |\mathbf{H}</em>{sd}^{(m)}|<em>F^2 s_s^{(m,k)} + \tilde{w}</em>{sd}^{(m,k)}, k = 1...K_s$</td>
</tr>
<tr>
<td>$U_{sr_q}^{(m)} = \mathbf{H}<em>{sr_q}^{(m)} S_r^{(m)} + Z</em>{sr_q}^{(m)}, m = 1...M_s$</td>
<td>$\tilde{u}<em>{sr_q}^{(m,k)} = |\mathbf{H}</em>{sr_q}^{(m)}|<em>F^2 s_r^{(m,k)} + \tilde{w}</em>{sr_q}^{(m,k)}, k = 1...K_s$</td>
</tr>
<tr>
<td>$U_{r_q d}^{(n_q)} = \mathbf{H}<em>{r_q d}^{(n_q)} S</em>{r_q}^{(n_q)} + Z_{r_q d}^{(n_q)}, n_q = 1...M_q$</td>
<td>$\tilde{u}<em>{r_q d}^{(n_q,r_q)} = |\mathbf{H}</em>{r_q d}^{(n_q)}|<em>F^2 s</em>{r_q}^{(n_q,r_q)} + \tilde{w}<em>{r_q d}^{(n_q,r_q)}, r_q = 1...K</em>{r_q}$</td>
</tr>
</tbody>
</table>

As a result, the set of samples which are transmitted from $r_q$ to $d$ is specified by $S_{r_q} = \{\{\beta_{r_q}^{(m,k)} r_q^{(m,k)}\} K_s | m,s \}$. The members of $S_{r_q}$, which now act as information symbols for $r_q$, are encoded by $M_{r_q} = B/K_{r_q}$ code matrices where $K_{r_q}$ is the number of information symbols per code matrix generated by $r_q$. (Here we assume that $B$ is also an integer multiple of $K_{r_q}$.) The relay nodes transmit the generated code matrices to the destination node through orthogonal channels. After transmission on the $n_q$th code matrix from $r_q$, denoted by $S_{r_q}^{(n_q,m)}$, $n_q = 1...M_{r_q}$, the destination receives $U_{r_q d}^{(n_q)}$ as shown in Table I and generates $K_{r_q}$ decision statistics denoted by $\tilde{u}_{r_q d}^{(n_q,r_q)}$ in the table, each of which is actually a noisy version of one member of $S_{r_q}$,
The mapping between pairs \((i, j)\) and \((n_q, v_q)\) in the expression given for \(\tilde{u}_{r,q}(n_q, v_q)\), is specified by \(i = \left\lfloor (n_q-1)K_q + v_q \right\rfloor\) and \(j = (n_q - 1)K_q + v_q - (i - 1)K_s\) in which \(\left\lfloor x \right\rfloor\) is equal to the smallest integer not less than \(x\). According to this mapping, we can easily show that when \(n_q\) varies between 1 and \(M_q\) and \(v_q\) varies between 1 and \(K_q\), \(i\) varies between 1 and \(M_s\) and \(j\) varies between 1 and \(K_s\), which verifies that there is a one-to-one mapping between the generated statistics \(\tilde{u}_{r,q}(n_q, v_q)\), \(n_q = 1...M_q\), \(v_q = 1...K_q\), and the members of set \(S_q\).

If we substitute \(\tilde{u}_{r,s}(i,j)\) in the expression given for \(\tilde{u}_{r,q}(n_q, v_q)\) in Table I, from the given expression for \(\tilde{u}_{r,q}(i,j)\) in the table (after replacing \((i, j)\) with \((m, k)\)), an alternative formula for \(\tilde{u}_{r,q}(n_q, v_q)\) can be obtained as follows:

\[
\tilde{u}_{r,q}(n_q, v_q) = \|\mathbf{H}_{s,q}^{(i)}\|^2 + \|\mathbf{H}_{s,q}^{(j)}\|^2 + \sum_{i=1}^{K} \mathbf{H}_{s,q}^{(i)}\mathbf{H}_{s,q}^{(j)}(\beta_{r,q})^2,\]

in which \(\tilde{u}_{r,q}(n_q, v_q)\) is a white zero-mean Gaussian noise sample with variance \(\|\mathbf{H}_{s,q}^{(i)}\|^2 + \|\mathbf{H}_{s,q}^{(j)}\|^2 + \sum_{i=1}^{K} \mathbf{H}_{s,q}^{(i)}\mathbf{H}_{s,q}^{(j)}(\beta_{r,q})^2 + 1\).

Equation (4) shows that \(\tilde{u}_{r,q}(n_q, v_q)\) contains information about \(s_{(i,j)}\) which is one of the transmitted information symbols from the source node. Therefore, according to (4) and the expression given for \(\tilde{u}_{r,q}(i,j)\) in Table I, we can observe that for every transmitted symbol from the source node, the destination generates \(Q\) decision statistics from the relays and 1 decision statistics from \(s_r\) to estimate \(\Lambda_{m,k}\), which after some calculations can be represented as

\[
\lambda_{m,k} = \left( A_m + \sum_{q=1}^{Q} B_m C_{n_q}(\beta_{r,q})^2 + 1 \right) \frac{\varepsilon_q}{N_0},
\]

where parameters \(\gamma_{r,q}, \gamma_{s,q}\), and \(\gamma_{r,s}\) are defined as:

\[
\gamma_{r,q} = A_m \varepsilon_q / N_0, \quad \gamma_{s,q} = B_m \varepsilon_q / N_0, \quad \gamma_{r,s} = C_n \varepsilon_q / N_0.
\]

III. Symbol Error Rate Analysis

Clearly, the error rate performance of the described multi-antenna cooperative system depends on the distribution of the signal to noise ratio in the decision variable \(\Lambda_{m,k}\). This SNR is referred to the post-detection SNR and to find an expression for it, we substitute \(\tilde{u}_{r,s}(m,k)\) from its definition in Table I and \(\tilde{u}_{r,q}(r_q, g_q)\) from (4), (after replacing \((n_q, v_q)\) with \((f_q, g_q)\)), into (6). Therefore \(\Lambda_{m,k}\) can be written alternatively as

\[
\Lambda_{m,k} = A_m + \sum_{q=1}^{Q} B_m C_{n_q}(\beta_{r,q})^2 + 1,\]

where \(A_m = \|\mathbf{H}_{r,q}^{(m)}\|^2, B_m = \|\mathbf{H}_{r,q}^{(m)}\|^2,\) and \(C_n = \|\mathbf{H}_{r,q}^{(m)}\|^2.\)

The post-detection SNR is defined as the ratio of the energy of the signal component to the energy of the noise component in the decision variable \(\Lambda_{m,k}\), which after some calculations can be represented as

\[
\lambda_{m,k} = \left( A_m + \sum_{q=1}^{Q} B_m C_{n_q}(\beta_{r,q})^2 + 1 \right) \frac{\varepsilon_q}{N_0},\]

where \(\gamma_{r,q}, \gamma_{s,q}\), and \(\gamma_{r,s}\) are defined as:

\[
\gamma_{r,q} = A_m \varepsilon_q / N_0, \quad \gamma_{s,q} = B_m \varepsilon_q / N_0, \quad \gamma_{r,s} = C_n \varepsilon_q / N_0.
\]

Since these parameters are dependent on the realization of the channel matrices, the post-detection SNR for each information symbol is represented as a random variable (RV) which according to (9) can be expressed as follows:

\[
\chi = \sum_{q=1}^{Q} \frac{\chi_{s,q}}{\chi_{r,s} + \chi_{r,q} + 1},\]

where \(\chi_{s,q}\) and \(\chi_{r,q}\) are the RVs corresponding to the parameters \(\gamma_{s,q}, \gamma_{r,q}\), and \(\gamma_{r,s}\), respectively. It can be verified that \(\chi_{s,q} \sim \mathcal{N}(d_{ab}, \gamma_{s,q})\) in which \(d_{ab} = N_a N_b\) and \(\gamma_{s,q} = \Omega_{ab} \varepsilon_q / N_0\) and for \(a = s\) or \(r_q\) and \(b = r_q\) or \(d\). Here, the notation \(\mathcal{N}(x, y)\) means that the RV \(X\) has gamma distribution with parameters \(u\) and \(v\) [10].

The SER can be expressed using Marcum’s Q-function as

\[
P_e = \int_{0}^{\infty} Q \left( \sqrt{\kappa_{mod} x} \right) p_X(x)dx,
\]

where \(p_X(x)\) is the probability density function (PDF) of the RV \(\chi\) and \(\kappa_{mod}\) is a constant which depends on the modulation type. As shown in [11], if the derivatives of \(p_X(x)\) up to order \(b - 1\) are null, at high SNRs (11) can be approximated using the McLaurin series of \(p_X(x)\) as follows:

\[
P_e = \frac{1}{b+1} \sum_{k=0}^{b} \frac{(2i - 1)}{(2i + 1)(2i + 1)!} \frac{\partial^b p_X(x)}{\partial x^b}(0).
\]

Therefore, in order to calculate \(P_e\), we need to determine the value of \(b\) so that \(\frac{\partial^b p_X(x)}{\partial x^b}(0) = 0\) for \(t < b\) but \(\frac{\partial^b p_X(x)}{\partial x^b}(0) \neq 0\), and also find the corresponding value of \(\frac{\partial^b p_X(x)}{\partial x^b}(0)\). For this purpose we first approximate \(\chi\) at high SNRs as follows:

\[
\chi \approx \chi_{s,q} + \sum_{q=1}^{Q} \frac{\chi_{s,q}}{\chi_{r,q} + \chi_{r,s}},
\]

and then we use the following proposition which is a generalization of Proposition I in [9].
Proposition 1: If we define a RV $W$ as $W = X + \sum_{q=1}^{Q} Y_q Z_q / (Y_q + Z_q)$ in which the RVs $X$, $Y_q$ and $Z_q$ $q = 1...Q$ are mutually independent and have gamma distributions so that $X \sim g(a, x)$, $Y_q \sim g(b_q, y_q)$, and $Z_q \sim g(c_q, z_q)$, where $a$, $b_q$, and $c_q$ are integers, and if we represent the PDF of the RV $W$ by $p_W(w)$, then all derivatives of $p_W(w)$ up to order $R - 1$ are null, where $R = a - 1 + \sum_{q=1}^{Q} \min(b_q, c_q)$, and
\[
\frac{\partial^R p_W}{\partial x^R}(0) = \frac{1}{\pi} \prod_{q=1}^{Q} \left[ \frac{1}{y_q} \cdot 1(b_q \leq c_q) + \frac{1}{z_q} \cdot 1(c_q \leq b_q) \right],
\]
in which function $1(c)$ is equal to 1 if $c$ is a true statement and is equal to 0 if $c$ is a false statement.

Proof: See Appendix.

Using the above proposition, we can establish that:
\[
\frac{\partial^b p_W}{\partial x^b}(0) = \frac{1}{\gamma_{sd}} \prod_{q=1}^{Q} \left[ \frac{1}{\gamma_{srq}} \cdot 1(d_{srq} \leq d_{srq}) \right] + \frac{1}{\gamma_{rd}} \cdot 1(d_{rd} \leq d_{rd}),
\]
(14)
where $b = d_{sd} - 1 + \sum_{q=1}^{Q} \min(d_{srq}, d_{rd})$. After substituting $\frac{\partial^b p_W}{\partial x^b}(0)$ from (14) into (12), and replacing the parameters $\gamma_{sd}$, $\gamma_{srq}$, and $\gamma_{rd}$ in the achieved expression with their definitions earlier, we arrive at the following expression for $P_e$:
\[
P_e \approx \frac{\prod_{i=1}^{g_d} (2i - 1)}{2^d d!} \frac{1}{\Omega_{sd}^d} \prod_{q=1}^{Q} \left[ \frac{1}{\Omega_{srq}^d} \cdot 1(d_{srq} \leq d_{srq}) \right] + \frac{1}{\Omega_{rd}^d} \cdot 1(d_{rd} \leq d_{rd})
\]
(15)
in which $g_d = b + 1 = N_s N_d + \sum_{q=1}^{Q} \min(N_s N_{r_q}, N_r N_d)$.

From (15), we observe that the diversity order of the cooperative system is $g_d$. If we look at the expression given for $g_d$, we can see that $g_d$ is equal to the summation of $Q + 1$ terms, each of which represents the achieved diversity order of one of the existing links between $s$ and $d$ (direct or relayed). The term $N_s N_d$ in this summation represents the achieved diversity order of the direct MIMO link between $s$ and $d$, which is the maximum achievable spatial diversity order of this link [7]. Similarly, the term $\min(N_s N_{r_q}, N_r N_d)$ represents the achieved diversity order of the multi-antenna relayed link $s \rightarrow r_q \rightarrow d$, which is equal to the minimum value of the maximum achievable diversity orders of $s \rightarrow r_q$ and $r_q \rightarrow d$ links. Intuitively, we can say that this value is the maximum achievable diversity order of the two hops $s \rightarrow r_q \rightarrow d$. Hence, we can observe that the proposed cooperative strategy can provide the maximum diversity order.

IV. Simulation Results

To evaluate the SER performance of the proposed cooperative system, we present some simulation results. In our simulations, we assume that each of the source and relay nodes has two transmit antennas and uses the Alamouti space-time code, and that the destination terminal has a single receiver antenna. For comparison, we also consider a similar configuration but with single antenna relay nodes which was considered in [6]. The results are obtained for QPSK modulation and we also assume $\Omega_{ab} = 1$ for all links.

Fig. 2 shows the SER vs. SNR for the various cooperative wireless systems. For fair comparison between these systems, we define SNR as $R(\varepsilon_a / N_0)$ in which $R$ is the summation of the number of antennas of the source and all relay nodes. The figure shows the theoretical SER values obtained using (15) for the cooperative systems with double antenna relay nodes as well as those obtained through computer simulations for all the systems. Good agreement between theoretical expectations and simulation results is observed for the double antenna relay systems. As we expect, when the number of relay nodes or the number of antennas of each node increases, the SER performance improves. For example, at SER equal to $10^{-5}$, increasing the number of single antenna relay nodes from 1 to 2 and 2 to 4 provides approximately 3 and 2.5 dB gain, respectively. We can also observe that at the same SER, shifting from one double antenna relay to two double antenna relays, provides a 2.5 dB gain.

Another fact is that at high SNRs, the SER performance of the cooperative system with two single antenna relays is the same as that of the cooperative system with one double antenna relay. The same is true for the system with 4 single antenna relays and the system with 2 double antenna relays. Therefore, we can replace two single antenna relay nodes with one double antenna relay (or vice versa) and still have the same SER performance. We can also replace 4 single antenna relay terminals with 2 double antenna relays while the SER remains unchanged. As a general rule and based on (15), we can say that the cooperative systems with different antenna configurations can have the same SER performance provided that $g_d$ is the same for all of them. Hence, for example, the systems with 4 single antenna or 2 double antenna relay terminals can also be replaced with a system in which the source has 3 antennas, the destination has one antenna and one relay node with 3 antennas is used ($g_d=6$ for all of them).
V. CONCLUSIONS

In this paper, we proposed a cooperative model for multi-antenna wireless networks based on OSTBCs. We found an approximate formula for the SER of the proposed system at high SNRs which determines the diversity order of the system. Based on the achieved results we observed that the proposed cooperative strategy can provide maximum diversity order and various multi-antenna cooperative systems with different structures could have the same SER performance.

REFERENCES


APPENDIX

Using the initial value theorem of Laplace transform we have:

\[
\frac{\partial q_{PV}}{\partial \omega R}(0) = \lim_{s \rightarrow -\infty} s^{R+1} \Psi_W(s),
\]

which the variables \(q_{\ast}, q_{vq}, q = 1...Q\) are defined so that \(R = \eta_x + Q + \sum_{q=1}^{Q} \eta_{vq}\). If we use the initial value theorem of Laplace transform, (17) can be expressed alternatively as

\[
\frac{\partial^q PV}{\partial \omega^q Q}(0) = \frac{\partial^q PV}{\partial x^{q} \omega^{q}}(0) \times \prod_{q=1}^{Q} \frac{\partial^q PV}{\partial v^{q} \omega^{q}}(0),
\]

where \(PV(x)\) represents the PDF of the RV \(X\) and \(PV_{vq}(v_q)\) is the PDF of the RV \(V_{q}\). Since \(X\) has a gamma distribution with parameters \(a, \bar{x}\), \(PV(x)\) is defined as [10]:

\[
PV(x) = \int_{D\{u,v\}} pv_{q}(u)pz_{q}(v) \ du \ dv,
\]

where \(PV_{vq}(u)\) is the PDF of the RV \(Y_q\) and \(pz_{q}(v)\) is the PDF of the RV \(Z_{q}\). In (20), we used the assumption that \(Y_q\) and \(Z_{q}\) are mutually independent.

The first derivative of \(PV_{vq}(x)\) is actually equal to \(pv_{q}(x)\) and thus for finding \(\frac{\partial^{v_q} PV}{\partial x^{v_q} \omega^{v_q}}(0)\), we need to calculate the \((v_{q} + 1)^{th}\) derivative of \(PV_{vq}(x)\) evaluated at \(x = 0\). To this aim, we can divide the region \(D\) (20) into five region and write \(PV_{vq}(x)\) as \(PV_{vq}(x) = I_1 + I_2 + I_3 + I_4 + I_5\), in which \(I_1 = \int_0^\infty \sum_{d=0}^{\infty} p_{X}(x)p_{Z}(v) \ du \ dv\), \(I_2 = \int_0^\infty \sum_{d=0}^{\infty} p_{X}(x)p_{Z}(v) \ du \ dv\), \(I_3 = \int_0^\infty \sum_{d=0}^{\infty} p_{X}(x)p_{Z}(v) \ du \ dv\), \(I_4 = \int_0^\infty \sum_{d=0}^{\infty} p_{X}(x)p_{Z}(v) \ du \ dv\), and \(I_5 = \int_0^\infty \sum_{d=0}^{\infty} p_{X}(x)p_{Z}(v) \ du \ dv\). In order to calculate the value of the \((v_{q} + 1)^{th}\) derivative of \(PV_{vq}(x)\) at \(x = 0\), we need to find the value of the corresponding derivative of each of these five integrals at \(x = 0\), and add these values together. Following these steps, we can find out that:

\[
\frac{\partial^{v_q} PV}{\partial x^{v_q} \omega^{v_q}}(0) = \frac{\partial^{v_q} PV}{\partial x^{v_q} \omega^{v_q}}(0) + \frac{\partial^{v_q} PV}{\partial v^{v_q} \omega^{v_q}}(0),
\]

Since \(Y_q \sim g(b_q, \bar{y}_q)\) and \(Z_q \sim g(c_q, \bar{z}_q)\) and we assumed that \(b_q, c_q\) are integers, \(\frac{\partial^{v_q} PV}{\partial x^{v_q} \omega^{v_q}}(0)\) is equal to 0 for \(\eta_{vq} < b_q - 1\) and it is equal to \(1/\bar{y}^b\) for \(\eta_{vq} = b_q - 1\). In addition \(\frac{\partial^{v_q} PV}{\partial v^{v_q} \omega^{v_q}}(0)\) is equal to 0 for \(\eta_{vq} < c_q - 1\) and it is equal to \(1/\bar{z}^c\) for \(\eta_{vq} = c_q - 1\). Taking into account these facts and the aforementioned fact that \(\frac{\partial^{v_q} PV}{\partial x^{v_q} \omega^{v_q}}(0)\) is zero for \(\eta_{vq} < a - 1\) and it is equal to \(1/\bar{x}^a\) for \(\eta_{vq} = a - 1\), and considering (18), we can conclude that:

\[
\frac{\partial^q PW}{\partial x^q R}(0) = \frac{1}{\bar{x}^a} + \sum_{q=1}^{Q} \left( \frac{1}{\bar{y}^b} \cdot \min(b_q, c_q) + \frac{1}{\bar{z}^c} \cdot \min(b_q, c_q) \right),
\]

\[
R = a - 1 + \sum_{q=1}^{Q} \min(b_q, c_q).
\]